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SYMMETRIC AUCTIONS: FINDING NUMERICAL SOLUTIONS BY USING AUCTION SOLVER

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Abstract

This essay theoretically investigates symmetric types of auctions and provides simulation for the different types of statistical distributions that bidders follow in order to determine the maximum bid, Expected revenue for the auctioneer, Expected payment by bidders, Expected utility of the bidders, probability that item is unallocated. In addition, the subject of interest of this paper are three types of auctions: FPA (first price auction, Dutch type, SPA (Second price auction, English auction, Vickrey auction), APA (all-pay auctions)

Introduction

Symmetric auctions are widely used in practice. These are types of auctions where bidders submit sealed bids, and the highest bid is winning.³¹ The payments are determined as an anonymous function of bids, (Deb & Pai 2016). Symmetric auctions are having rules, which are anonymous and non-discriminatory, which is why they are so popular in practice, despite the fact that they do not achieve seller's revenue maximisation when buyers are heterogeneous. For instance, first price auctions (FPA) are more profitable than the second price auctions (SPA) in a case of risk averse buyers and when risk aversion coefficient is higher the difference is larger. So it is plausible that FPA is more profitable than SPA, (Klempeper 1999). Klempeper asserts that in second price or ascending bid auction, winner pays a price set by a runner-up (second best), and by revenue equivalence, must bid the expectation of this price in a first price auction. So the price is fixed in FPA and random but with the same mean in SPA. This means that the price is riskier in SPA. Therefore, risk averse seller prefers first-price auction to a second sealed-bid auction, and prefers the second price sealed-bid auction to an ascending open auction.

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Subject to being greater than reservation bid.

31) Subject to being greater than reservation bid.

Interactions between markets and institutions-The case of auctions

McAfee and McMillan (1987), define auctions as market institutions with an explicit set of rules that are determining resource allocation and prices on a basis of bids from the market participants.

On the other hand institutions are rules that define conditions by which market messages (offers, demands, and accepts) are being traded, and lead to allocation of resources and process, (Smith2001). If in the economy there exist n agents, i.e. $i = 1, 2, \dots, n$ and each of them chooses a message μ_i , then the allocation of the resources x_i towards agent i is defined as :

Equation 1

$$x_i = \eta (m_1, m_2, \dots, m_i, \dots, m_n)$$

In the previous expression η are the rules of allocation. Rules of allocation are institutionally defined rules. As an example one can take auctions and their rules. Let's suppose that bidder i outs a highest bid. In such case following applies:

Equation 2

$$x_i = \eta (m_i > m_1 > m_n) = 1, x_k = 0; \forall k \neq i$$

In this case m_i is the highest bid, and agent $i = 1$ puts a highest bid. Henceforth, agent 1 will win the auction. At first when agent 1 will make the bid, nobody knows who will win the lot (An item or set of items for sale in an auction). But the other on-site bidders know, that none of the other k agents will not like to raise the bid, so that the rules of the institution tell us that $x_1 = 1$ and $x_k = 0$. Some of the most successful internet auction houses such as eBay and Amazon are using second price auctions, by which bidder with highest bid will win the lot., but pays a price equal to the small increment (growth) on top of a second highest bid. On-site bidder can offer and "reserve price", which is a maximum price and which is used to compete the others with prices. Some auction houses such as eBay have fixed closing time, while on Amazon closing time is continued as long as few minutes pass without bid (ten minutes on Amazon). In the eBay type of auction model there exist n bidders, $i = 1, 2, \dots, n$. Minimal offer is m (reserve price), and the least increment is ζ , which is assumed to be constant.³² Reserve price should be higher than the current price, and the last reserve price ("Hammer price") established by the last bidder, i.e. bidder cannot lower its last reserve price which was set, (Ockenfels&Roth 2003). Highest reserve price is not revealed. Auction history shows the last price and shows the time when price was put by the bidder. Auctioneer can bid at any time $t \in [0, 1] \cup 1$. Reaction time of other bid of second bidder $t' < 1$. Earliest time of reaction is set with t_n and $t_n > t'$, and is close to one, and so $t' < t_n < 1$. On the eBay auctions there are no dominant strategies. Bid caller (auctioneer), in this auctions can pay some value v_j , distributed by some known distribution.³³ Auctioneer who will win the auction with some price p , earn the difference between the price he likes to pay, and the price that actually wins the auction $v_j - p$, auctioneer who does not won, does not gets zero. In the eBay auctions that are *second price of the auctioneer* type of auctions not with private value auctions v_j , there are no dominant strategies $v_j > m + \zeta$. The number of auctioneers is 2, $i = 2$, and by assumption if $i > 2$, other auctioneers do not auction. There is no dominant strategy for the player j who has value $v_j > m + \zeta$, which is best response for any strategy of the player j , which is the best response to any strategy of the player i . Let's suppose that the strategy, i , is to set minimal bid m when $t = 0$, i.e. at the beginning of the auction, and to bid furthermore always when there is highest bid, but if he is not the auctioneer with the highest bid than he will set bid $b > v_j + \zeta$. Contrary to this strategy, best response of the player j is not to bid at any time peri-

32) i.e., $\frac{\partial \zeta}{\partial p} = c$, where, c is constant.

33) This strategy for value which the player could bid immediately applies only for the beginning of the auction, or in any time between the beginning and the end of the auction.

od, $t < 1$, and not to bid v_j when $t = 1$. Result from this strategy for player j , $p(v_j - m - \zeta) > 0$, which means that no other strategy when $t = 1$ cannot contribute to higher result to j . But, if we assume that the strategy of the player i is not to bid in any of the times offered, than the strategy of the player i who answers the bid only when $t = 1$, that will provide him the expected result $p(v_j - m) < v_j - m$, which is a result that j could achieve from the strategies offered v_j when $t = 0$, or at any given period of time when $t < 1$. Best strategy for the player i is $p(v_j - m - \zeta) > 0$, which is a strategy of player j , because best strategy of j was not to bid if $t \neq 0$. Consequently, this is a proof that there is no dominant strategy. In a standard auction model of the eBay institution, it has been specified minimal opening bid, but not the price, which is a seller's reserve price. Seller can choose public bid but also additional reserve price below which he cannot sell. In the previous example reserve price is lower so that the post-sale negotiation be avoided.³⁴

First price auction (sealed first bid auction strategically equivalent to Dutch auctions)

In this auction scheme each buyer submits a bid $b \geq 0$ for the object. The bids are sealed, and no bidder has information about the bids of other opponents. The object is allocated to the bidder with the highest bid. Highest bidder pays the bid; other bidders pay nothing. Variance in the *Dutch* auctions is

For the Dutch type of auctions variance of price is:

Equation 3

$$\begin{aligned}\sigma_{pa}^2 &= \int (p_d - \bar{p}_d)^2 dp(v) = \int_0^1 \left(\frac{n-1}{n} v - \frac{n-1}{n+1} \right)^2 n v^{n-1} dv \\ &= \int_0^1 \frac{n^2-1}{n^2} v - \frac{(n-1)(n+1)}{(n+1)^2} n v^{n-1} dv = \frac{(n-1)(n+1)}{n^2} v \\ &\quad - \frac{(n-1)(n+1)}{(n+1)^2} n v^{n-1} \Big|_0^1 = \frac{\frac{n-1}{n} - \frac{n-1}{n+1}}{n+1} \\ &= \frac{(n-1)(n+1)(1-n)}{n^2(n+1)} = \frac{(n-1)(1-n)}{n^2}\end{aligned}$$

In the Dutch model of auction, buyers gain is $v - p = v - \left[\frac{n-1}{n} \right] v = \frac{v}{n}$; where v is the highest drawn value, range of possible gains is from 0 to $\frac{1}{n}$. While, in the English type of auction gain could vary from 0 to 1, Vickrey (1961)

Vickrey type of auction (sealed-bid)

This auction scheme is different in the one presented in (Vickrey1961), which is used by Google and Yahoo in their *on-line* advertisement programs. Vickrey (1961) auction scheme, is different from the type of scheme of *proxy auctions* on eBay, by the fact that bids are not closed, i.e. *current highest bid*³⁵ is always public. In Vickrey type of auction, bidders submit their bids sealed, without knowing other members in the auction bids, and in this type of auction highest bid wins, but the price paid is second highest bid. This result is Pareto

34) On this way there is incentive agent not to bid above its own bid if he is the highest bid caller because of the fear reserve price not to exceed his initial bid.

35) Current highest bid is equal to the second best bid plus increment change, i.e. $m + \zeta$

optimal. This auction is efficient because the winner is the auctioneer for whom the lot has highest value. Achievement of Pareto optimal result on imperfect markets is difficult. Market imperfection in Vickrey model is in the fact that buyers and sellers are very few in number to ignore consequences of their actions on price but on the other hand there are not enough participants in the market in overt or tacit (collusions) market actions. Vickrey in his model describes one marketing agency, through which all sales of goods and procurements must be done. If we imagine homogenous rectangular distribution of the auction in limits 0 to 1, and even though number of players is $i > 1, 2, \dots, n$, from which for all of them there exist corresponding value in the distribution v_i and are bidding one price, while probability for first $n - 1$ players to draw corresponding value for them is $[0, v]$ while the n -th player would draw corresponding value between $[v, v + dv] = v^{n-1} dv$, this value is from the value $dp_1(v) = nv^{n-1}$. Price of every bid is given as $p_d = \left(\frac{n-1}{n}\right)v$. Expected price in this case is given with the following expression:

Equation 4

$$\bar{p}_d = \int_0^1 p dp_1(v) dv = \int_0^1 \left(\frac{n-1}{n}\right) v nv^{n-1} dv = \frac{(n-1)(nv^{n-1})}{n} dv = \frac{n-1}{n+1}$$

Previous case is when one has definite integral, i.e. distribution goes from 0 to 1, and the result applies if $n > -1$, if there is unlimited distribution than the result is:

Equation 5

$$\begin{aligned} \bar{p}_d &= \int p dp_1(v) dv = \int \frac{(n-1)(nv^{n-1})}{n} dv = \\ &= \frac{(n-1)n}{n} \int v v^{n-1} dv = \frac{(n-1)n}{n} \int v^n dv = \frac{(n-1)n}{n} \frac{v^{n+1}}{n+1} + C = \frac{(n-1)v^{n+1}}{n+1} + C \end{aligned}$$

Probability that the second best price will be realized is $\frac{n}{n+1}$, and the probability that this price is in the interval between $[v, v + dv] n(n-1)v^{n-2}(1-v)dv$, meaning that the average price is:

Equation 6

$$\bar{p}_e = \int v dp_2(v) = \int_0^1 n(n-1)(v^{n-1} - v^n) dv = (n-1) - \frac{n(n-1)}{n+1} = \frac{n-1}{n+1}$$

Again expected value of the bidders is the expected value of the excess $\frac{n}{n+1}$, and the value of lot won by the successful bidder $\frac{n}{n+1}$. This applies for progressive type of auctions, and for Dutch auctions.³⁶ But the price variance is different in both types of auctions.³⁷ For the SPA (second price auction theory) variance is

Equation 7

$$\begin{aligned} \sigma_{e,d}^2 &= \int_0^1 \left(v - \frac{n-1}{n+1}\right)^2 n(n-1)(v^{n-2} - v^{n-1}) dv \\ &= \left(v - \frac{n-1}{n+1}\right)^2 n(n-1)(v^{n-2} - v^{n-1}) \Big|_0^1 = \left(1 - \frac{n-1}{n+1}\right)^2 n(n-1) \\ &= \left(1 - 2\frac{n-1}{n+1} + \frac{n^2 - 2n + 1}{n^2 + 1}\right) n(n-1) \\ &= \frac{[2(n+1)(n+1)^2 + n^2 2(n+1) - 4n(n+1) + 2(n+1)]n(n-1)}{2(n+1)(n+1)^2} \\ &= \frac{n + 2[(n+1)^2 + n^2 - 2n + 1]n(n-1)}{(n+2)(n+1)^2} \\ &= \frac{(n+2)n(n-1)[(n+1)^2 - (n-1)^2]}{(n+2)(n+1)^2} = \frac{2(n-1)}{(n+2)(n+1)^2} \end{aligned}$$

36) English auctions are such type where it is begun with lowest (reserve prices) and it is going to higher price. English auctions are of open type.

37) Dutch type of auction. is a type of auction, that begins with highest bid that decreases until some auctioneer does accept the price, or accepts proposed reserve price.

Variance in the gain is even larger from Dutch to English model of auction. *Revenue equivalence theorem* is a central topic in the auction theory. This theory confirms that if there are n risk neutral agents, that do independent and personal evaluation of some auction good, and valuation follows cumulative distribution $F(v)$, which is ascending probability distribution of a continuous set of choices (v, v) . Then every auction mechanism (every institution auction), in which lot will be allocated towards the agent for which it has highest value v , and every agent with a valuation of good v has utility 0, generates exact same revenue, which lead every bidder to make the same payment. Mechanism of lot allocation taken into consideration here is following: expected value of lot for the agent i is $u_i(v)$, while the probability that the agent i will win the auction is $P_i(v)$, in the equilibrium of the auction (when the lot is sold), and afterwards it follows the agents strategy v , this strategy is given by a following expression:

Equation 8

$$u_i(\bar{v}) = \bar{v}P_i(\bar{v}) - E(\text{payment } \bar{v}_i)$$

This means that the utility of agent i is dependent on the average value by which he values lot, which is multiplied with the probability that he will win the auction minus the average value of agents i payment, which he will do for the selected lot. In the equilibrium:

Equation 9

$$u_i(v) \geq u_i(\bar{v}) + (v - \bar{v})P_i(\bar{v})$$

v type agent, values the lot with $v - \bar{v}$, more than agents \bar{v} lot valuation. Inequality applies because the deviation must be non-profit, i.e. $v - \bar{v} \geq 0$, here one it can be written that

Equation 10

$$v = v + dv$$

Where dv is some marginal value, and v value is larger than \bar{v} for some marginal value. If in the previous expression $u_i(v) \geq u_i(\bar{v}) + (v - \bar{v})P_i(\bar{v})$ one replaces \bar{v} with $v = v + dv$, one could get :

Equation 11

$$u_i(v) \geq u_i(v + dv) + (v - v + dv)P_i(v + dv)$$

With the rearrangement one can get:

Equation 12

$$u_i(v) \geq u_i(v) + dvP_i(v)$$

Value of the utility for the agent i for v valuation, one can get as a sum of utility $u_i(v)$, plus the sum from probability that lot x will be allocated to him, which is equal to $P_i(v)$, i.e.:

Equation 13

$$u_i(v_i) = u_i(v) + \int_v^{v_i} P_i(v)dv$$

Because there are two possible outcomes of the auction, agent wins the lot and agent does not win the lot, $(0,1)$, i.e. as it is said in the theorem at the beginning (v, v) , meaning that the agent of type v will never won the game, and his utility will be zero $u_i(v) = 0$. Furthermore, every player has equal probability that will win the lot in the two type of auctions (English and Dutch) $P_i(v_i)$. From the equation, $u_i(v) = vP_i(v) - E(\text{payment } v_i)$, for the i -th agent we have :

Equation 14

$$u_i(v_i) = v_i P_i(v_i) - E(\text{payment } v_i)$$

This means that expected payoff in the two mechanisms (v, v) , is equal and amounts to $E(\text{payment } v_i)$.

Symmetric model case

This case applies when coefficient of risk aversion is zero. Whether one chooses Constant relative risk aversion (CRRA),³⁹ or Constant absolute risk aversion (CARA),³⁸ if the risk aversion coefficients are zero, and reservation price $(r = 0)$ revenue equivalence holds. In this case expected revenue $E(R)$, from the auction to the auctioneer is the same in the First price auction (*Dutch auction*) and in the Second price auction (*English auction*, Vickrey type of auction). So in the case where revenue equivalence holds (risk neutral case), utility to the bidders is:

Equation 15

$$U(c) = c$$

In the previous expression c represents consumption, and in the risk neutral case such as this bid functions (symmetric equilibrium bid functions) are:

Equation 16

$$b(v) = \left(\frac{n-1}{n} \right) v$$

In the case of a pure strategy equilibrium, where player 1 (bidder) reveals, bidder's valuation is common knowledge, so the rival bidders can anticipate bid of player 1. This bid will be denoted as $v_0 > 0$, unless $v_1 = 0$. Best response of the closest rival (i -th bidder) to the bidder 1 rival will be to bid as in an auction with $n - 1$ bidders, and a common knowledge reserve price b_0 which will be equal to v_0 . The equilibrium bidding strategy with $n - 1$ bidders and a reservation price of b_0 and reservation value $v > b_0$, is given as:⁴⁰

Equation 17

$$b(v) = b_0 + \frac{1}{n-1} \int_{b_0}^v F^{n-1}(x) dx$$

$b_0 < v < 1$ or zero otherwise. In the previous integral s is a signal;

In the previous expression $b(v_i)$ represents buyers i valuation of the object bid that wins the object and in such a case v_i represents bidders i reservation value. And the bidding strategy is strictly increasing function of his reservation value, v represents the private values of the bidders, x represents private information that buyer i has obtained about the objective monetary valuation of the object $F^{n-1}(x)$ is buyers probability of winning the lot, while $F^{n-1}(v)$ represents a highest bid. In such a situation winning bid in general is given as: $b(v) = p(v)/F^{n-1}(v)$. In the previous expression $p(v)$ represents expected payment which is equal to: $p(v) = v F^{n-1}(v) - \int_{b_0}^v F^{n-1}(x) dx$. If $v_* = v$ than in FPA and SPA, expected revenue is the same. That is only the case if reservation price is zero. Otherwise the seller will always announce reservation price strictly greater than his personal valuation i.e. $v_* > v$, so: $v_* = v_0 + 1 - \dots$. Where v_* is a minimum auction reservation value and below this reservation value is not worthwhile bidding, $F(v_*)$ is a probability function

38) This is also known as Arrow-Pratt measure of relative risk aversion. See in (Carl&Blume1994)

39) This is also known as Arrow-Pratt measure of absolute risk aversion.

40) See (Riley & Samuelson 1981)

that bidder with valuation $F'(v_*)$, will win the lot, and F' is a first derivative of the probability function. By rewriting and simplifying, we will find bidding function first, we will solve the integral:

Equation 18

$$\int_r^v F^{n-1}(s) ds = \int_r^v s^{n-1} ds$$

Since signal $s=v$, Indefinite integral is $\int v^{n-1} dv = \frac{v^n}{n} + C$. So to solve definite integral one may apply:

Equation 19

$$\left(\frac{v^n}{n}\right) | (v=v) = \frac{v^n}{n} \text{ and } \left(\frac{v^n}{n}\right) | (v=r) = \frac{r^n}{n}$$

In general solution to the definite integral is :

Equation 20

$$\int_r^v \frac{v^n}{n} dv = \frac{v^n}{n} - \frac{r^n}{n}$$

Bidding function in the risk averse case where $a=0$, and r is set is given as:

Equation 21

$$b(v_i) = v - \frac{1}{v^{n-1}} * \left[\frac{v^n}{n} - \frac{r^n}{n} \right]$$

Or in general case for non-uniform distributions bidding function in FPA according to Krishna (2010) is given as:

Equation 22

$$\beta(v) = x - \int_0^v \frac{F(y)}{F(x)} dy$$

And $F(y) = 1 - F(x)$ ($F(x)$ is a CDF of a function), x signal are drawn from private values distribution v so $x_i = v_i$

Now in the CRRA case, RRA coefficient can take values between zero and 1, utility is given as $(c) = c^{1-a}$, where a is the coefficient of RRA and $a \in (0, 1)$. Then the Nash-equilibrium bidding functions will have the following form:

Equation 23

$$b(v_i) = v - \frac{1}{\frac{n-1}{n-1-a}} \int_r^v F^{1-a}(s) ds$$

And $b_0 < v < 1$ or zero otherwise.

One general Nash equilibrium bidding strategy (CRRA case and where $r=0$) in FPA is given as:

Equation 24

$$b(v_i) = \frac{n-1}{n-1+a} (v_i)$$

And where $r>0$ i.e. auction is set with the reserve price bidding function is given as:

Equation 25

$$b(v_i) = v - \frac{1}{F^{1-a}(v)} \int_r^v F^{\frac{n-1}{1-a}}(s) ds$$

To solve this we find the indefinite integral solution ($F(v)=v$) which is :

Equation 26

$$\int v^{\frac{n-1}{1-a}}(x) dx = \frac{v^{\frac{a-n}{1-a}+1}}{(a-n)} (a-1) + C$$

Namely if $n = \frac{n-1}{1-a}$ than $\int v^n dv = \frac{v^{n+1}}{n+1} dv + C; n \neq -1$

Equation 27

$$\int v^n dv = \frac{v^{\frac{n-1}{1-a}+1}}{\frac{n-1}{1-a}+1} + C = \frac{v^{\frac{a-n}{1-a}}}{(a-n)} * (a-1) + C$$

Now to evaluate integral at the endpoints

Equation 28

$$\frac{v^{\frac{a-n}{1-a}}}{(a-n)} * (a-1) | (v=v) = \frac{v^{\frac{a-n}{1-a}}}{(a-n)} * (a-1)$$

$$\frac{v^{\frac{a-n}{1-a}}}{(a-n)} * (a-1) | (v=r) = \frac{r^{\frac{a-n}{1-a}}}{(a-n)} * (a-1)$$

$$\text{So } \int_r^v v^{\frac{n-1}{1-a}} dv = \frac{v^{\frac{a-n}{1-a}}}{(a-n)} * (a-1) - \frac{r^{\frac{a-n}{1-a}}}{(a-n)} * (a-1)$$

So the general bid function in this case when $a>0$ and $r>0$ is given s:

Equation 29

$$b(v_i) = v - \frac{1}{F^{\frac{n-1}{1-a}}(v)} * \left[\frac{v^{\frac{a-n}{1-a}}}{(a-n)} * (a-1) - \frac{r^{\frac{a-n}{1-a}}}{(a-n)} * (a-1) \right]$$

If the coefficient of risk aversion is CARA (Constant absolute risk aversion), ie. $u(c) = 1 - e^{-ac}$, where $a > 0$, than the bidding function is given as:

Equation 30

$$b(v_i) = v + \frac{1}{a} \ln \left(1 - \frac{e^{-av}}{F^{a-1}(v)} \int_{e^{ar}}^{e^{av}} [F(a^{-1} \ln w)]^{N-1} dw \right)$$

Or when reserve price and absolute coefficient of risk aversion is set:

Equation 31

$$b(v_i) = v + \frac{1}{a} \ln\left(1 - \frac{e^{-av}}{v^{n-1}} * \frac{1}{2a} * (e^{2av} - e^{2ar}) * \log(v)^{n-1}\right)$$

Where w presents wealth of the bidder. Approximately wealth is approximated by his valuation of the object which is subject to bidding at the auction.

All-pay auctions

In all –pay auctions every bidder pays their bid whether they win or not. In all pay auctions Nash-equilibria (in the case of risk neutrality) mixing strategies of the bidders means that expected sellers revenue is equal to the common value of the prize,⁴¹ are approximately given by the expression:

Equation 32

$$F(b) = \left(\frac{b}{100}\right)^{\frac{1}{n-1}}$$

Previous expression is valid for all b , if $0 \leq b \leq 100$, in that expression $F(b)$ also denotes c.d.f of the equilibrium bidding strategy ; see (Gneezy&Smorodinsky 2006)

In general, in risk neutral case and there is no reserve price set bidding function for All-pay auctions is given as:

Equation 33

$$b_{(all-pay)} = v^{n-1} \left[\left(\frac{n-1}{n} \right) * v \right]$$

Or $b_{(all-pay)} = v^{n-1} * \beta(v)$ where $\beta(v)$ denotes FPA bid-First price auction bid. If the reserve price is set i.e. $r>0$ and bidders are risk neutral $a=0$ than the bidding function in APA auctions is given as:

Equation 34

$$b_{(all-pay, r>0)} = v^{n-1} * v - \int_r^v v^{n-1} dv$$

i.e.

Equation 35

$$b_{(all-pay, r>0)} = v^{n-1} * v - \left[\frac{v^n}{n} - \frac{r^n}{n} \right]$$

in the case where CRRA coefficient is being set bid function in APA auctions takes the following form:

Equation 36

$$b_{(all-pay, r>0)} = v^{n-1} * \left[\frac{n-1}{n-1+a} (v_i) \right]$$

If $r>0$ and $a>0$ i.e. there is RRA coefficient set and reserve price at the same

41) See Baye et al (1996)

Equation 37

$$b(v_i) = v^{n-1} * \left[v - \frac{1}{F^{\frac{n-1}{1-a}}(v)} * \left[\frac{v^{\frac{a-n}{a-1}}}{(a-n)} * (a-1) - \frac{v^{\frac{a-n}{a-1}}}{(a-n)} * (a-1) \right] \right]$$

In CARA case

Equation 38

$$b(v_i) = v^{n-1} * \left[v + \frac{1}{a} \ln \left(1 - \frac{e^{-av}}{v^{n-1}} * \frac{1}{2a} * (e^{2av} - e^{2av}) * \log(v)^{n-1} \right) \right]$$

In general in the previous case with risk averse bidders, bidding functions are approximations the problem of optimization is as follows (according to the software creator Richard M.Katzwer), consider an all pay auction where n bidders draw IPV from some distribution with cdf on $[v_l, v_h]$. And let $\beta: [v_l, v_h] \rightarrow \mathbb{R}$ denote the symmetric Nash bid function. Bidders are risk averse with utility index u and initial wealth W . A bidder tries to maximize:

Equation 39

$$\max_b u(W + v - b)[F(\beta^{-1}(b))]^{n-1} + u(W - b)(1 - [F(\beta^{-1}(b))]^{n-1})$$

The First order necessary condition (FONC) here is given as:

Equation 40

$$0 = u'(W + v - b)[F(\beta^{-1}(b))]^{n-1}$$

This is a complicated ODE to solve.

Expected revenue (expected payoffs in auctions)

In the first price auctions expected revenue for the auctioneer for every bid is given as expected payment multiplied by the number of bidders n , i.e.:

Equation 41

is the expected payoff, where v_i are independent private values, and revenue is given as:

Equation 42

$$E(R, b_i, v_i) = E(p, b_i, v_i) * N$$

When CRRA coefficient is being set i.e. when bidders are not risk neutral the corresponding expected pay-offs (and corresponding revenue) of the bidders are given as:

Equation 43

$$E(R, b_i, v_i) = v_i * \frac{(b_i)^{n-1}}{\left(\frac{n-1}{n-1+\alpha}\right)^{n-1}}$$

Where α is CRRA coefficient in general when bids for the corresponding probabilities are known (calculated) the expected payoffs (expected revenues for every bidder) are given as:

Equation 44

$$E(R, b_i, v_i) = v_i * \frac{(b_i)^{n-1}}{\left(\frac{n-1}{n-1+\alpha}\right)^{n-1}}$$

Equation 45

When reserve price is set in SPA auction, revenue function below reserve price looks like this:

Equation 46

While when independent private values exceed reserve price the functional form of revenue CDF looks same as in the case with no reserve price.

In all pay auctions in general case expected payoffs of the bidders are given as

Equation 47

Expected revenue in APA is given as:

Equation 48

From the revenue one need to find the private values function first and that is :

Equation 49

$$v = \left(\frac{\beta(v)}{\left(\frac{2}{3}\right)} \right)^{\left(\frac{1}{3}\right)}$$

Results from auction solver

Next, here are presented the results obtained by use of auction solver. Richard M.Katzwer wrote this software. In the First table, we present the results obtained for the risk neutral case and in the second Table, we provide results for the case of risk neutral bidders.⁴³

Table 1 Risk neutral bidders, with reservation price zero, number of bidders=3

Risk averse case CRRA=0.4 ;CARA=4.0 and reservation price=0.4; number of bidders n=3						
Type of private values distribution	Auction type	maximum bid (right endpoint of bid distribution \bar{b})	Expected revenue for the auctioneer ($E[R] = E[m] * \bar{b}$)	Expected payment by bidders $E[m]$	Expected utility to bidders $E[u]$	probability item is unallocated (all bids below reserve) P_0
Standard normal (CRRA=0.4,r=0.4)	FPA(first price auction , Dutch type)	0.7346317	0.5233480	0.1744493	0.1028037	0.0943964
	SPA(Second price auction, English auction, Vickrey auction)	1.0000000	0.4798612	0.1599537	0.1131171	0.0943964
	APA(all-pay auctions)	0.7346317	0.0819876	0.0273292		
Standard normal (CARA=4.0,r=0.4)	FPA(first price auction , Dutch type)	0.7263772	0.5102678	0.1700893	0.1534737	0.0943964
	SPA(Second price auction, English auction, Vickrey auction)	1.0000000	0.4798612	0.1599537	0.1534737	0.0943964
	APA(all-pay auctions)	0.7263772	0.0756754	0.0252251		
Log normal (CRRA=0.4,r=0.4)	FPA(first price auction , Dutch type)	0.8649665	0.7540914	0.2513638	0.0828988	0.0002989
	SPA(Second price auction, English auction, Vickrey auction)	1.0000000	0.7077322	0.2359107	0.0966421	0.0002989
	APA(all-pay auctions)	0.8649665	0.1718317	0.0572772		
Log normal (CARA=4.0,r=0.4)	FPA(first price auction , Dutch type)	0.8384763	0.7290578	0.2430193	0.1297647	0.0002989
	SPA(Second price auction, English auction, Vickrey auction)	1.0000000	0.7077322	0.2359107	0.1297647	0.0002989
	APA(all-pay auctions)	0.8384763	0.1559933	0.0519978		
Beta (CRRA=0.4,r=0.4)	FPA(first price auction , Dutch type)	0.7735836	0.5761506	0.1920502	0.1036646	0.0640000
	SPA(Second price auction, English auction, Vickrey auction)	1.0000000	0.5256000	0.1752000	0.1156296	0.0640000
	APA(all-pay auctions)	0.7735836	0.1036405	0.0345468		

43) CRRA and CARA are being used as measures of neutrality or aversity of bidders.

Beta (CARA=4.0,r=0.4)	FPA(first price auction , Dutch type	0.7598812	0.5603228	0.1867743	0.1567281	0.0640000
	SPA(Second price auction, English auction, Vickrey auction)	1.0000000	0.5256000	0.1752000	0.1567281	0.0640000
	APA(all-pay auctions)	0.7598812	0.0952758	0.0317586		
Uniform (CRRA=0.4,r=0.4)	FPA(first price auction , Dutch type	0.7735836	0.5761506	0.1920502	0.1042956	0.0640000
	SPA(Second price auction, English auction, Vickrey auction)	1.0000000	0.5256000	0.1752000	0.1163464	0.0640000
	APA(all-pay auctions)	0.7735836	0.1036405	0.0345468		
Uniform (CARA=4.0,r=0.4)	FPA(first price auction , Dutch type	0.7598812	0.5603228	0.1867743	0.1576778	0.0640000
	SPA(Second price auction, English auction, Vickrey auction)	1.0000000	0.5256000	0.1752000	0.1576778	0.0640000

Next, in Table 2 are presented the results in the risk averse bidders case.

Table 2 Risk averse bidders (CRRA=4.0,CARA=4.0), with reservation price 0.4,number of bidders=3

Triangular (CRRA=0.4,r=0.4)	APA(all-pay auctions)	0.7598812	0.0952758	0.0317586		
	FPA(first price auction , Dutch type	0.6901554	0.5442205	0.1814068	0.0892794	0.0327680
	SPA(Second price auction, English auction, Vickrey auction)	1.0000000	0.5077239	0.1692413	0.0997395	0.0327680
Triangular (CARA=4.0,r=0.4)	APA(all-pay auctions)	0.6901554	0.0679635	0.0226545		
	FPA(first price auction , Dutch type	0.6722348	0.5252554	0.1750851	0.1349865	0.0327680
	SPA(Second price auction, English auction, Vickrey auction)	1.0000000	0.5077239	0.1692413	0.1349865	0.0327680
Kumaraswamy (CRRA=0.4,r=0.4)	APA(all-pay auctions)	0.6722348	0.0584859	0.0194953		
	FPA(first price auction , Dutch type	0.7372686	0.5892777	0.1964259	0.0925610	0.0255160
	SPA(Second price auction, English auction, Vickrey auction)	1.0000000	0.5456206	0.1818735	0.1046092	0.0255160
Kumaraswamy (CARA=4.0,r=0.4)	APA(all-pay auctions)	0.7372686	0.0912166	0.0304055		
	FPA(first price auction , Dutch type	0.7165197	0.5682720	0.1894240	0.1420239	0.0255160
	SPA(Second price auction, English auction, Vickrey auction)	1.0000000	0.5456206	0.1818735	0.1420239	0.0255160
Gamma (CRRA=0.4,r=0.4)	APA(all-pay auctions)	0.7165197	0.0799332	0.0266444		
	FPA(first price auction , Dutch type	0.5756602	0.2727609	0.0909203	0.0719515	0.4180305
	SPA(Second price auction, English auction, Vickrey auction)	1.0000000	0.2592612	0.0864204	0.0748882	0.4180305
Gamma (CARA=4.0,r=0.4)	APA(all-pay auctions)	0.5756602	0.0203123	0.0067708		
	FPA(first price auction , Dutch type	0.5922903	0.2713597	0.0904532	0.1018825	0.4156601
	SPA(Second price auction, English auction, Vickrey auction)	1.0000000	0.2593647	0.0864549	0.1018825	0.4156601
Weibull (CRRA=0.4,r=0.4)	APA(all-pay auctions)	0.5922903	0.0208872	0.0069624		
	FPA(first price auction , Dutch type	0.7011976	0.4704638	0.1568213	0.1001409	0.1418659
	SPA(Second price auction, English auction, Vickrey auction)	1.0000000	0.4335800	0.1445267	0.1086913	0.1418659

	FPA(first price auction , Dutch type	0.7011976	0.0651847	0.0217282		
Weibull (CARA=4.0,r=0.4)	FPA(first price auction , Dutch type	0.6992714	0.4607301	0.1535767	0.1473960	0.1418659
	SPA(Second price auction, English auction, Vickrey auction)	1.0000000	0.4335800	0.1445267	0.1473960	0.1418659
	FPA(first price auction , Dutch type	0.6992714	0.0610630	0.0203543		
Power (CRR=0.4,r=0.4)	FPA(first price auction , Dutch type	0.6575730	0.3854263	0.1284754	0.0932573	0.2529822
	SPA(Second price auction, English auction, Vickrey auction)	1.0000000	0.3578528	0.1192843	0.0992482	0.2529822
	FPA(first price auction , Dutch type	0.6575730	0.0460285	0.0153428		
Power (CARA=4.0,r=0.4)	FPA(first price auction , Dutch type	0.6666767	0.3811101	0.1270367	0.1344171	0.2529822
	SPA(Second price auction, English auction, Vickrey auction)	1.0000000	0.3578528	0.1192843	0.1344171	0.2529822
	FPA(first price auction , Dutch type	0.6666767	0.0451887	0.0150629		
Reverse Power (CRR=0.4,r=0.4)	FPA(first price auction , Dutch type	0.9139690	0.7795748	0.2598583	1.8465012	0.0114520
	SPA(Second price auction, English auction, Vickrey auction)	1.0000000	0.7123353	0.2374451	Infinity	0.0114520
	FPA(first price auction , Dutch type	0.9139690	0.2064959	0.0688320		
Reverse Power (CARA=4.0,r=0.4)	FPA(first price auction , Dutch type	0.8877422	0.7553769	0.2517923	2.9094572	0.0114520
	SPA(Second price auction, English auction, Vickrey auction)	1.0000000	0.7123353	0.2374451	Infinity	0.0114520
	FPA(first price auction , Dutch type	0.8877422	0.1904224	0.0634741		
Exponential (CRR=0.4,r=0.4)	FPA(first price auction , Dutch type	0.7384851	0.5237630	0.1745877	0.1034777	0.0977777
	SPA(Second price auction, English auction, Vickrey auction)	1.0000000	0.4796241	0.1598747	0.1138245	0.0977777
	FPA(first price auction , Dutch type	0.7384851	0.0836569	0.0278856		
Exponential (CARA=4.0,r=0.4)	FPA(first price auction , Dutch type	0.7305809	0.5111237	0.1703746	0.1543483	0.0977777
	SPA(Second price auction, English auction, Vickrey auction)	1.0000000	0.4796241	0.1598747	0.1543483	0.0977777
	FPA(first price auction , Dutch type	0.7305809	0.0775855	0.0258618		

Conclusion

Hugo Sonnenschein, in his 1983 inaugural Nancy Schwartz Memorial Lecture, argued that is very important to understand how incentives work, i.e. how to create institutional arrangements that induce individuals to behave in certain way so that certain outcome prevails. In 1990s and later, economists have been recognized to the design of several auction like mechanisms such as: US FCC spectrum auctions, 3G auctions in Europe, the allocation of property rights on land at the airports, etc. So, the market design, or market architecture depends on a game theory, in particular games of incomplete information, but most important insights come from auction theory, Menezes, Flavio, M. , Monteiro ,Paulo K.,(2008). Auction mechanism is also very important since it is market clearing mechanism that equates supply and demand. In auctions price formation is explicit, i.e. the rules that determine price formation are well known to all parties involved. This paper provided a theoretical insight how bids are formed in symmetric auctions case, in the three types of auctions FPA, SPA, APA.

Appendix 1. Distributions in Auction solver software

Standard Normal distribution function

In this case Φ is cumulative distribution, than standard normal function with CDF

$(\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-z^2/2} dz$ and $z = \frac{x-\mu}{\sigma}$ z is standardized normal value ;in this case of standard normal distribution $z=x$) given as :

$$F(x) = \frac{\Phi\left(\frac{x-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}$$

In our case we used standard normal distribution function where $\omega_L = 0$ and $\omega_H = 1$ and $\mu = 0$ and $\sigma = 1$. So this gives :

$$F(x) = \frac{\Phi\left(\frac{x-0}{1}\right) - \Phi\left(\frac{a-0}{1}\right)}{\Phi\left(\frac{b-0}{1}\right) - \Phi\left(\frac{a-0}{1}\right)} = \frac{\Phi(x) - \Phi(a)}{\Phi(b) - \Phi(a)}$$

Also ω_L is the lowest boundary of support where $F=0$, and ω_H is the highest boundary of support where $F=1$. Also, $F(x)$ this is a cumulative function of a distribution .This CDF should be differentiable and strictly increasing function on some interval (ω_L, ω_H) is a subset form points in the interval $[0, \infty]$. And, $f(x)$ is the probability density function of a distribution, that is derivative of cumulative distribution function $f = F'$.

$$f(x) = \frac{\frac{1}{\sigma} \phi\left(\frac{x-\mu}{\sigma}\right)}{\Phi\left(\frac{\omega-\mu}{\sigma}\right) - \Phi\left(\frac{-\mu}{\sigma}\right)}$$

since in our case $\mu = 0$ and $\sigma = 1$ we can simplify the previous expression as:

$$f(x) = \phi(x)$$

in our case $\mu = 0$ but $\sigma = 0.5$ so the last expression can be simplified as:

$$F(x) = \frac{\int_a^x \frac{1}{z * 0.5\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ln z}{0.5}\right)^2\right] dz}{\int_a^b \frac{1}{z * 0.5\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ln z}{0.5}\right)^2\right] dz}$$

And the probability density function given as:

$$f(x) = \frac{\frac{1}{x\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2\right]}{\int_a^b \frac{1}{z\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ln z - \mu}{\sigma}\right)^2\right] dz} \text{ or in our case:}$$

$$f(x) = \frac{\frac{1}{x\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ln x}{0.5}\right)^2\right]}{\int_a^b \frac{1}{z * 0.5\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ln z}{0.5}\right)^2\right] dz} \text{ and } z=x \text{ also could be simplified in the last expression.}$$

Beta distribution

In the Beta distribution it should be $\alpha, \beta > 0$ so in our case $\alpha = \beta = 1$ (uniform distribution) our distribution support values between 0 and 1, support=[0.000,1.000]. The Beta function is a power function on x and its reflection $1-x$, and it is defined as:

$$B(a, b) = \int_0^1 x^{a-1} (1-x)^{b-1} dx; a > 0, b > 0, 0 \leq x \leq 1$$

And CDF of this function can be written as:

$$F(x) = \frac{1}{B(a, b)} \int_{\omega_L}^{v(x)} x^{a-1} (1-x)^{b-1} dx; \text{ where } v(x) = \frac{x - \omega_L}{\omega_H - \omega_L}$$

PDF of the function is given as:

$$f(x) = \frac{1}{\omega_H - \omega_L} \frac{v(x)^{a-1} (1-v(x))^{b-1}}{B(a, b)}$$

With $\alpha = \beta = 1$ our Beta distribution turns into standard uniform distribution

$$\frac{1}{B(a, b)} x^{a-1} (1-x)^{b-1}$$

Uniform distribution

In the uniform distribution $\omega_H > \omega_L \geq 0$ and in our case this distributions is supported in the range [0,1]. Cumulative distribution function for uniform distribution is given as:

$$F(x) = \frac{x - \omega_L}{\omega_H - \omega_L}$$

Probability distribution function is given as:

$$f(x) = \frac{1}{\omega_H - \omega_L}$$

Inverse function is given as:

$$F^{-1}(x) = \omega_L + x[\omega_H - \omega_L]$$

Triangular distribution

In our case in the triangular distribution parameters were set as follows: $a = 0$, $b = 1$, and $c = 0.5$. Requirement here is that $a \leq c \leq b$. Also $\psi \in [0, 1]$ this is parameter of uniformity (vertex-common endpoint of two or more rays of line segments). In our case $\psi = 0$.

Cumulative distribution function here is:

$$F(X) = \psi x + (1 - \psi) \frac{(x-a)^2}{(b-a)(c-a)} \quad \text{for } x \in [a, c] \text{ or}$$

$$F(X) = \psi x + (1 - \psi) \left[1 - \frac{(b-x)^2}{(b-a)(b-c)} \right] \quad \text{for } x \in [c, b]$$

In our case since $a = 0$, $b = 1$, and $c = 0.5$ this distribution is a distribution of $x = \frac{(x_1 + x_2)}{2}$, x_1 and x_2 are two independent random variables. So previous expression in our case would become:

$$F_{(0, \dots, 0.5)}(X) = \frac{(x)^2}{(0.5)} = 2x^2 \quad \text{if } 0 \leq x \leq 1/2 \quad \text{or}$$

$$F_{(0.5, 1 - 0.5)}(X) = 1 - \frac{(1-x)^2}{0.5} = (2x - 1)^2 \quad \text{and for the part } 1/2 \leq x \leq 1 \text{ it becomes}$$

$$F_{(0.5, \dots, 1)}(X) = 1 - \frac{(1-x)^2}{0.5} = 2x^2 - (2x - 1)^2$$

$$F(x) = \begin{cases} \frac{(x)^2}{(0.5)} = 2x^2 & \text{if } x \in (0, \dots, 0.5) \\ 2x^2 - (2x - 1)^2 & \text{if } x \in (0.5, \dots, 1) \end{cases}$$

$$4x - (4x - 4x^2)$$

Probability distribution function or PDF is a first derivative of the CDF so

$$f(x) = F'(x) = \begin{cases} 4x & \text{if } x \in (0, \dots, 0.5) \\ 4 - 4x & \text{if } x \in (0.5, \dots, 1) \end{cases}$$

Kumaraswamy or double bounded distribution

In our case for Kumaraswamy distribution lower and upper bound are $\omega_L = 0$ and $\omega_H = 1$ and

$\alpha = \beta = 2$. Requirement fulfilled here is that $\alpha, \beta > 0$, and $\omega_L, \omega_H \in \mathbb{R}_+$. The CDF of the Kumaraswamy distribution is defined as:

$$F(x) = 1 - (1 - x^\alpha)^\beta$$

$\Gamma(x) = \int_0^{\infty} x^{k-1} e^{-x} dx; k \in (0, \infty)$ (this is also called lower incomplete Gamma function)

CDF of Gamma distribution is given as:

$$F(X) = \frac{\Gamma(k, \frac{x}{b})}{\Gamma(k)} \quad x \in (0, \infty) \quad \text{or CDF is given as}$$

$$F(x) = \frac{\int_0^{\infty} x^{k-1} e^{-x} dx}{\Gamma(k)}$$

While the probability density function is given as :

$$f(x) = \frac{1}{\theta^k \Gamma(k)} x^{k-1} e^{-\frac{x}{\theta}}$$

Or in terms of shape parameter k and scale parameter b , PDF of Gamma distribution can be written as:

$$f(x) = \frac{1}{b^k \Gamma(k)} \exp \left[(k-1) \ln(x) - \frac{1}{b} x \right] \quad x \in (0, \infty) \quad \text{or if we simplify:}$$

$f(x) = \frac{1}{b^k \Gamma(k)} * x^{k-1} * e^{-\frac{x}{b}}$ in our case where $k=0.5$, $b=1$ the last expression would become:

$$\text{First } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} = \sqrt{3.1416} = 1.7725 \approx 1.8$$

$$f(x) = 1.8 * \frac{1}{\sqrt{x}} * e^{-x} = 1.8 * \frac{1}{\sqrt{x}} * \frac{1}{e^x} = \frac{1.8}{e^x * \sqrt{x}}$$

Weibull distribution

This is continuous probability distribution. In our case $\omega_L = 0$ and $\omega_H = 1$ and $k = 1$ which means that the failure rate is constant over time, and $\lambda = 1$ this is the scale parameter. And after the Rayleigh distribution $\lambda = \sqrt{2\sigma}$ since in our case $\lambda = 1\sqrt{2\sigma} = 1$ and $= 1/2$. Cumulative distribution function of Weibull distribution is given as:

$$F(x, k, \lambda) = 1 - e^{-\left(\frac{x}{\lambda}\right)^k} \quad \text{or alternatively}$$

$$F(x) = \frac{1 - \exp \left[-\left(\frac{x - \omega_L}{\lambda}\right)^k \right]}{1 - \exp \left[-\left(\frac{\omega_H - \omega_L}{\lambda}\right)^k \right]}$$

Probability density function is given as:

$$f(x) = \frac{\frac{k}{\lambda^k} (x - \omega_L)^{k-1} * \exp \left[-\left(\frac{x - \omega_L}{\lambda}\right)^k \right]}{1 - \exp \left[-\left(\frac{\omega_H - \omega_L}{\lambda}\right)^k \right]}$$

Uniform distribution

This distribution is $U(a, b)$ in our case $a=0$, $b=1$, this are minimal and maximal value. This distribution supports $b \geq a \geq 0$.

So CDF of a uniform distribution is:

$$F(x) = \frac{x-a}{b-a} \quad a \leq x \leq b$$

PDF of uniform distribution is given as:

$$f(x) = \frac{1}{b-a}$$

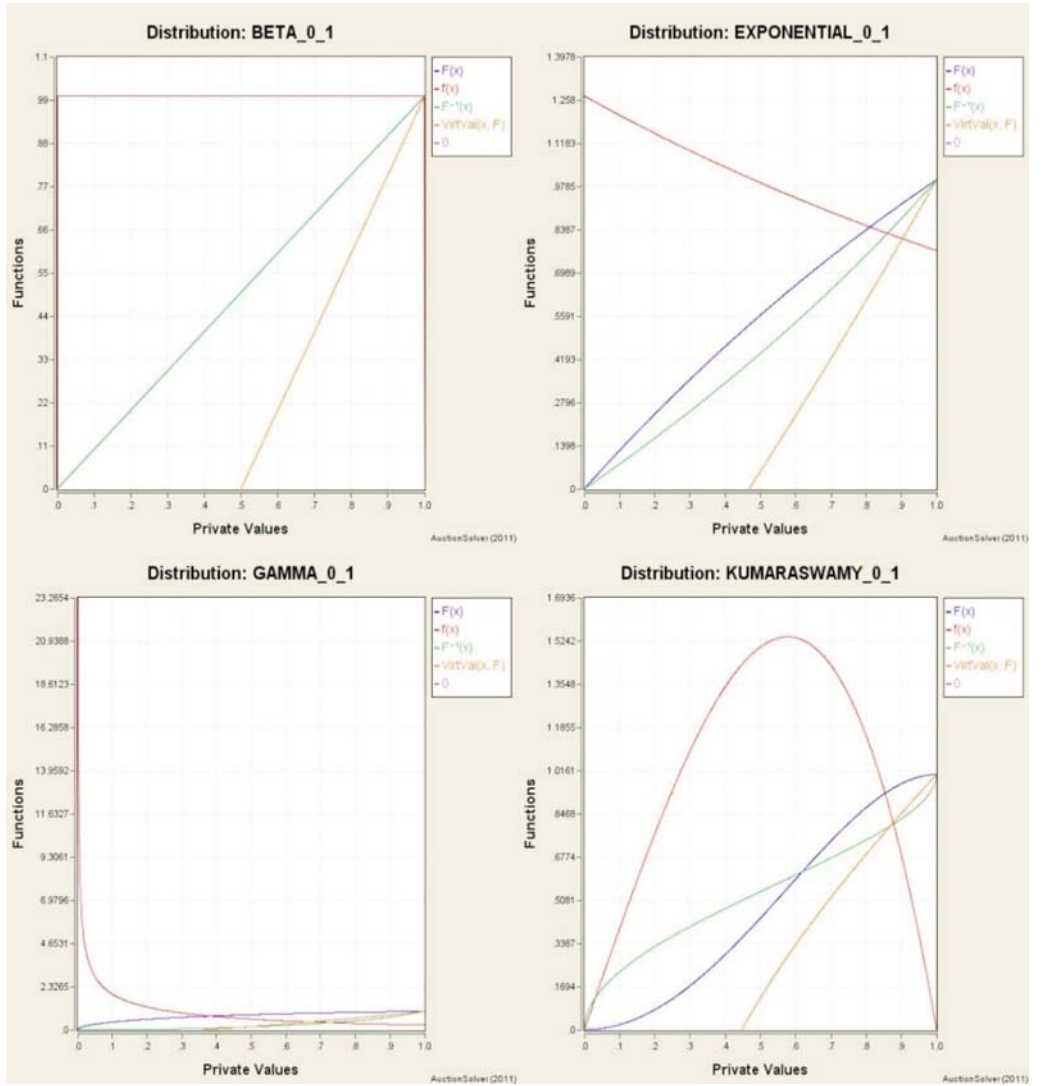
Power distribution

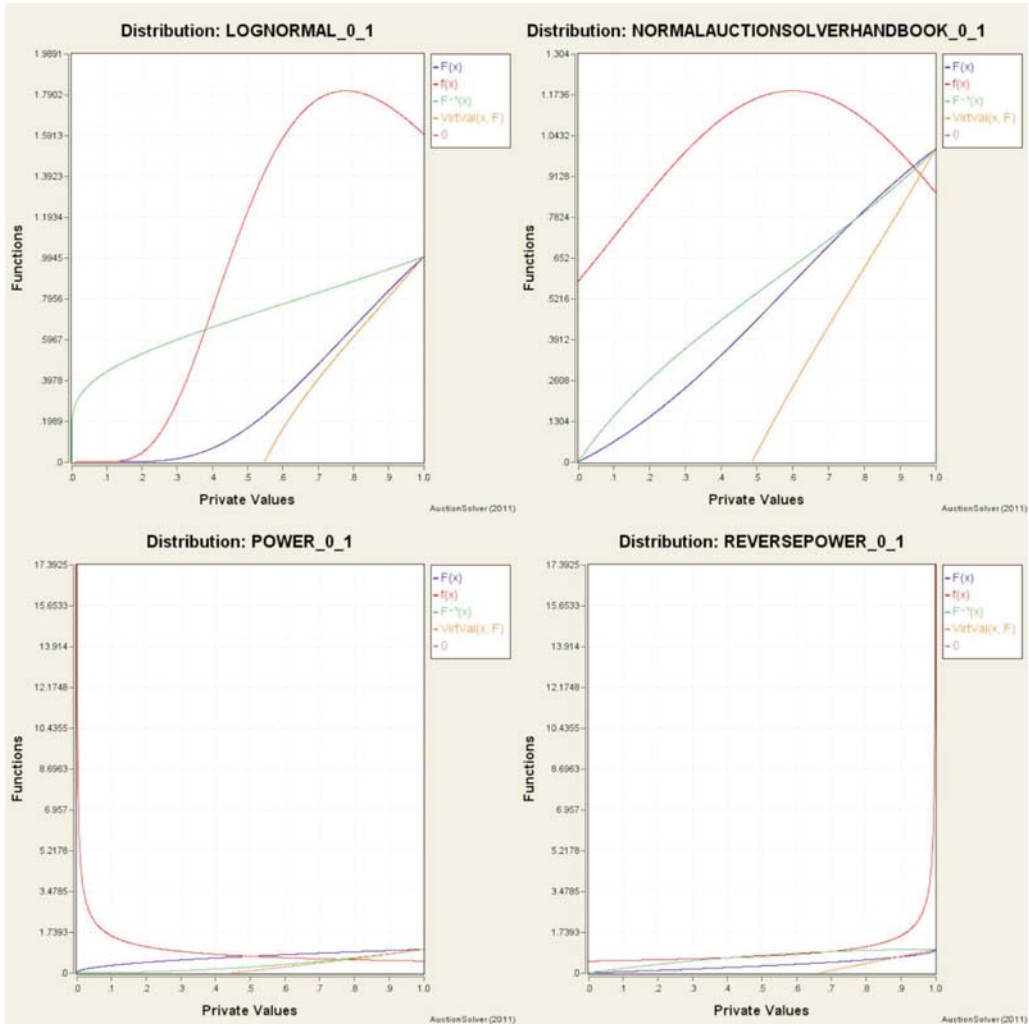
This is distribution that is frequently used to model income, financial variables and others. In our case $a = 0, b = 1, \alpha = 0.5$. CDF of this distribution¹ is given as:

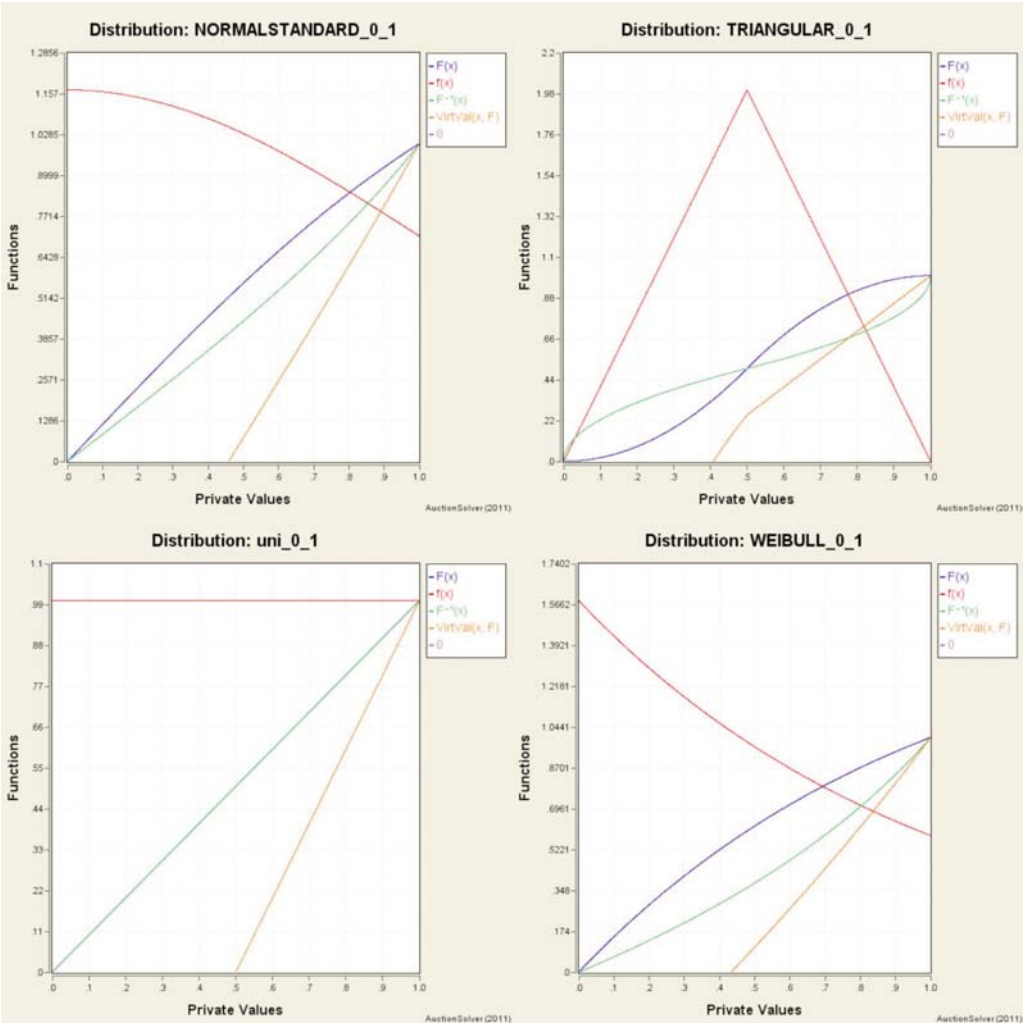
$$F(x) = \frac{\eta}{\alpha + 1} [(x + a + c)^{\alpha+1} - c^{\alpha+1}]$$

PDF of this distribution is given as:

$$f(x) = \eta (x + a + c)^{-\alpha}$$







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