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MODERN PORTFOLIO THEORY: IDENTIFICATION OF OPTIMAL PORTFOLIOS AND CAPITAL ASSET PRICING MODEL TEST

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Abstract

The tool we employ in this work is the well-known Modern Portfolio Theory (MPT), which forms the basis of virtually all quantitative portfolio management and theory today. Since its formulation half a century ago it has been seized on by the investment industry as a workable tool for investment and risk management, in particular because of its simplicity and intuitive appeal, and it remains one of the cornerstones in the foundation on which today's asset management industry rests. The MPT introduced the analysis of portfolios of investments by considering the expected return and risk of individual assets and, crucially, their interrelationship as measured by correlation. In MPT diversification plays an important role.
The Capital Asset Pricing Model (CAPM) relates the returns on individual assets or entire portfolios to the return on the market as a whole. In CAPM investors are compensated for taking systematic risk but not for taking specific risk. This is because specific risk can be diversified away by holding many different assets. We illustrate this concepts in an application on real market data. We use an optimization in order to find the optimal portfolios and then we test the CAPM.

Keywords: *Investments; portfolio performance; stock return; risk; volatility.*

Why measuring the capital account openness

In an asset allocation problem the investor, who can be the trader, or the fund manager, or the private investor, seeks the combination of securities that best suit their needs in an uncertain environment. In order to determine the optimum allocation, the investor needs to model, estimate, access and manage uncertainty. The most popular approach to asset allocation is the mean-variance framework, where the investor aims at maximizing the portfolio's expected return for a given level of variance and a given set of investment constraints. Under a few assumptions it is possible to estimate the market parameters that feed the model and

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then solve the ensuing optimization problem. This approach is highly intuitive. Sample estimates make sense only if the quantities to estimate are market invariants, i.e. if they display the same statistical behaviour independently across different periods. In equity like securities the returns are approximately market invariants: this is why the mean-variance approach is usually set in terms of returns.

We introduce in the next section some important theoretical concepts of asset management including a short literature review. The basis of any investment is the desire to obtain a return on that investment. The investor or asset manager must accept some amount of risk in order to obtain the return. In other words, the risk taken on by the investor is the price paid for the opportunity for a positive return, and the desired level of return thus determines the exact amount of risk taken on by the investor. This is a fundamental investment relationship, which investors must consider when deciding whether to invest in either a single asset or a portfolio of assets. The modern portfolio theory originally dates back to 1952, when Harry Markowitz published his article on what he called *'portfolio selection'.* In this article he established a framework for describing portfolios of assets in terms of the means on their returns, the variance of their returns, and the correlation between the returns on assets. For this reason the approach is also known as *mean-variance analysis*.

We test in section 3 some of the most important findings of Modern Portfolio Theory, the determination of the best efficient frontier and the Capital Asset Pricing Model test. We analyze in this section the stocks of ten international companies, part of the Standard and Poor stock index. We have selected a five years period, from March 31, 2006 to March 31, 2011. We end the work with some principal findings and conclusions, including some suggestions for further work.

2. Modern portfolio theory

2.1. Literature review

Portfolio theory took form as an academic field when Harry Markowitz published the article *'Portfolio Selection'* in 1952. Markowitz focuses on a portfolio as a whole; instead of security selection he discusses portfolio selection. Previously, little research concerning the mathematical relations within portfolios of assets had been carried out. Markowitz began from John Burr Williams' *Theory of Investment Value*. Williams (1938) claimed that the value of a security should be the same as the net present value of future dividends. Since the future dividends of most securities are unknown, Markowitz claimed that the value of a security should be the net present value of expected future returns. Markowitz claims that it is not enough to consider the characteristics of individual assets when forming a portfolio of financial securities. Investors should take into account the comovements represented by covariances of assets. If investors take covariances into consideration when forming portfolios, Markowitz argues that they can construct portfolios that generate higher expected return at the same level of risk or lower level of risk with the same level of expected return than portfolios ignoring the co-movements of asset returns. Risk, in Markowitz' model (as well as in many other quantitative financial models) is assessed as the variance of the portfolio. The variance of a portfolio in turn depends on the variance of the assets in the portfolio and on the covariances between its assets. Markowitz' mean variance portfolio model is the base on which much research within portfolio theory is performed. It is also from this model that the Black-Litterman model was developed. The Black-Litterman model builds on the Markowitz model and it is hence important to understands Markowitz' model.

Markowitz shows that investors under certain assumptions, *theoretically*, can build portfolios that maximize expected return given a specified level of risk, or minimize the risk given a level of expected return. The model is primarily a normative model. The objective for Markowitz has been not to explain how people select portfolios, but how they should select portfolios (Sharpe, 1967). Even before 1952 diversification was a well accepted strategy to lower the risk of a portfolio, without lowering the expected return, but until then, no thorough foundation existed to validate diversification. Markowitz' mean-variance portfolio model has remained to date the cornerstone of modern portfolio theory (Elton & Gruber, 1997).

2.2. Principal aspects

The basis of any investment is the desire to obtain a return on that investment. Since there is no such thing as a free lunch, the investor or asset manager must accept some amount of risk in order to obtain the return. In other words, the risk taken on by the investor is the price paid for the opportunity for a positive return, and the desired level of return thus determines the exact amount of risk taken on by the investor. This is a fundamental investment relationship, which investors must consider when deciding whether to invest in either a single asset or a portfolio of assets.

In order to properly evaluate investments we need a measure of return on those investments. We are not interested in asset prices, but rather in the returns on those assets. The return calculation presented in the next section is very simple. We apply it both to assets and to portfolios of assets. Financial risk is commonly quantified by some measure of variance of asset returns. We are interested in the variation of asset returns over time, and since we need some sort of reference point relative to which returns can be measured, we apply a measure of risk that relates every observation to the average or mean of all observations available. Variance is a simple measure of variation around an average. The standard deviation of returns is therefore defined as the square root of returns variance.

As mentioned above, the risk of a portfolio, quantified by its volatility, is heavily dependent on the exact nature and magnitude of the covariance or correlation between asset returns. If the returns on assets in the portfolio are correlated, there may exist opportunities for reducing the level of total portfolio risk by selecting appropriate assets and asset weights, in an attempt to offset individual asset risks against each other. In other words we attempt, in a structured manner, to exploit the fact that asset returns often move in somewhat consistent patterns relative to each other. Because the returns on assets are only very infrequently perfectly correlated, including several assets in a portfolio will tend to reduce overall portfolio risk. A very large number of stocks in a portfolio will entail larger transaction costs. The risk of adverse performance from a single stock increases with a large number of stocks. For these reasons the portfolio manager, in his quest for diversification, should attempt to exploit more precisely the characteristics of individual assets and asset classes.

In the quest for this type of non-naive or intelligent diversification, we thus need to establish an objective function that can guide our efforts towards making a selection of stocks that exploits each stock's particular characteristics in an efficient manner. We must specify precisely what is meant by the term 'efficiency' in a portfolio context. In general, efficiency is defined as the utilisation of resources in such a manner that the maximum output or gain is generated. Implicit in this definition is the quality of optimality. In a portfolio context we define efficiency as the maximum attainable return for a given level of volatility, or alternatively, the minimum attainable volatility for a given level of return. We designate efficient portfolios as those portfolios that cannot be improved upon in terms of the return versus risk trade-off. It is thus not possible to alter an efficient portfolio without paying a price in the form of lower return or higher volatility. As we shall see, the vast majority of attainable portfolios are not efficient in the strict mean-variance sense, which suggests that we can improve on them at no cost (in terms of return or volatility) by altering their composition.

The minimum-variance portfolio is the portfolio (that is, the combination of asset weights) that, given the particular return and risk characteristics of each asset, generates the lowest amount of risk achievable. In other words, the minimum variance portfolio specifies the asset weights that generate the lowest possible portfolio risk, without any additional constraints on the desired level of return or on the maximum or minimum extent to which an asset can enter into the portfolio. The efficient frontier is the line between the minimum-variance portfolio and the maximum variance portfolio that traces out all attainable portfolios (asset combinations) that produce optimal/efficient portfolios. In other words, the efficient frontier is the line in return/risk space that traces out all the portfolios for which we cannot obtain a higher level of return for a given level of risk, or alternatively for which we cannot obtain a lower level of risk for a given level of return. The portfolio that maximises return relative to risk (the Sharpe Ratio) is the portfolio that lies on the tangency point between the Asset Allocation Line and the efficient frontier.

One such theory, which has proven extremely robust and rugged since its birth in 1964, is the *Capital Asset Pricing Model* (CAPM). It basically proposes that an asset's return can be described completely by a combination of a market return and the asset's covariation with that market. Its logic is simple. The idea is that investors are compensated for taking on necessary risk but not for taking on unnecessary risk. It provides a framework for separating risk into necessary (systematic or market-related) risk, and unnecessary (unsystematic, asset-specific or residual) risk. The CAPM simply postulates that a linear relationship exists between the return on an asset and the return on the market, and that asset returns can thus be explained by a single factor, namely the market return.

3. Efficient frontier and Capital Asset Pricing Model test

In this section we analyze ten well known stocks, part of the U.S. market. We apply on these time series all the concepts we saw in the previous section. First, we determine the returns from the selected stock prices *(Total Return Data).* Using the returns, we calculate the variance-covariance and the correlation matrices useful for the efficient frontier construction. Then we choose 5 stocks of the 10 in order to determine the risky asset Efficient Frontier. We select the stocks so that to determine the "best" Efficient Frontier. We tried to choose stocks from completely different industry sectors in order to differentiate as much as possible my portfolio. All the selected stocks are part of the S&P 500 index. The period is from March 31, 2006 to March 31, 2011 (monthly observations). We can see in the table below first five observations of the selected stocks.

Table 1. Selected stocks (currency: US $).

HASBRO	*COCA COLA*	*JOHNSON & JOHNSON*	*HOST HOTELS & RESORTS*	*EXXON MOBIL*	*EDISON INTL*	*WHIRL POOL*	*TITANIUM METALS*	*BANK OF AMERICA CORP.*	*RED HAT*
tot return ind	**tot return ind**	**tot return ind**	**tot return ind**	**tot return ind**	**tot return ind**	**tot return ind**	**tot return ind**	**tot return ind**	**tot return ind**
13613,37	3425,67	4358,96	3325,16	9706,93	3945,54	1215,57	48,06	3619,08	51,28
13811,85	3413,33	4446,94	3075,29	9771,32	3999,74	1198,88	59,52	3571,33	41,68
14882,98	3521,1	4528,09	3071,45	12046,83	4000,97	1127,2	75,81	3592,89	44,06
14410,84	3450,74	4635,78	3198,63	11340,88	4308,64	1197,57	71,67	3429,32	41,91
13397,01	3597,31	4737,24	3248,84	10851,85	4504,71	1097,32	66,93	3502,41	41,3

Let's transform now prices in returns (in percentages) using: $R_t = \frac{P_t - P_{t-1}}{P_{t-1}} = \frac{P_t}{P_{t-1}} - 1$, where P_t is the price at time *t* and is the return at time *t*. In table 2, we have the return sample mean and standard deviation for each price series. The standard deviation, given as the square root of the variance, is a good estimator of the process volatility.

Table 2. Expected return and standard deviation.

Stocks	*HASBRO*	*COCA COLA*	*JOHNSON & JOHNSON*	*HOST HOTELS & RESORTS*	*EXXON MOBIL*
Exp. Return	0,0131	0,0088	0,0032	0,0075	0,0081
Standard Deviation	0,0767	0,0489	0,0401	0,1625	0,0589
Stocks	**EDISON INTL**	**WHIRLPOOL**	**TITANIUM METALS**	**BANK OF AMERICA CORP.**	**RED HAT**
Exp. Return	0,0054	0,0145	0,0377	0,0002	0,0243
Standard Deviation	0,057	0,1409	0,1689	0,1743	0,1479

As we can see, all the expected returns are positive. The corresponding parameters for the S&P index are 0,4 basis points (expected return) and 44 basis points (standard deviation). We have in table 3 the variance-covariance matrix for the ten selected stocks. We observe that all relations between assets are positive except 'Whirlpool' and 'Exxon Mobil'. The covariance between these two stocks is *-0,00072*.

Table 3. Covariance matrix.

Stocks	HASBRO	COCA COLA	JOHNSON & JOHNSON	HOST HOTELS & RESORTS	EXXON MOBIL	EDISON INTL	WHIRL POOL	TITANIUM METALS	BANK OF AMERICA CORP.	RED HAT
HASBRO	0,00588	0,00084	0,00131	0,00492	0,00116	0,00156	0,00466	0,0021	0,00528	0,00241
COCA COLA	0,00084	0,00239	0,00114	0,00166	0,00097	0,00084	0,00218	0,0017	0,00338	0,00077
JOHNSON & JOHNSON	0,00131	0,00114	0,00161	0,00203	0,00046	0,00098	0,00286	0,00074	0,00372	0,00104
HOST HOTELS & RESORTS	0,00492	0,00166	0,00203	0,02641	0,00048	0,00222	0,01492	0,01184	0,01515	0,00477
EXXON MOBIL	0,00116	0,00097	0,00046	0,00048	0,00347	0,00116	-0,00072	0,00262	0,00045	0,00042
EDISON INTL	0,00156	0,00084	0,00098	0,00222	0,00116	0,00325	0,00207	0,00252	0,003	0,00245
WHIRLPOOL	0,00466	0,00218	0,00286	0,01492	-0,00072	0,00207	0,01984	0,00485	0,01627	0,0046
TITANIUM METALS	0,0021	0,0017	0,00074	0,01184	0,00262	0,00252	0,00485	0,02852	0,00589	0,00686
BANK OF AMERICA CORP.	0,00528	0,00338	0,00372	0,01515	0,00045	0,003	0,01627	0,00589	0,0304	0,00514
RED HAT	0,00241	0,00077	0,00104	0,00477	0,00042	0,00245	0,0046	0,00686	0,00514	0,02188

We have calculated in table 4 the correlation matrix of the ten selected assets. As we can see from the table, almost all the stocks are positively correlated. The only negative correlation is between 'Whirlpool' and Exxon mobil', equal to -0,08879.

Table 4. Correlation matrix.

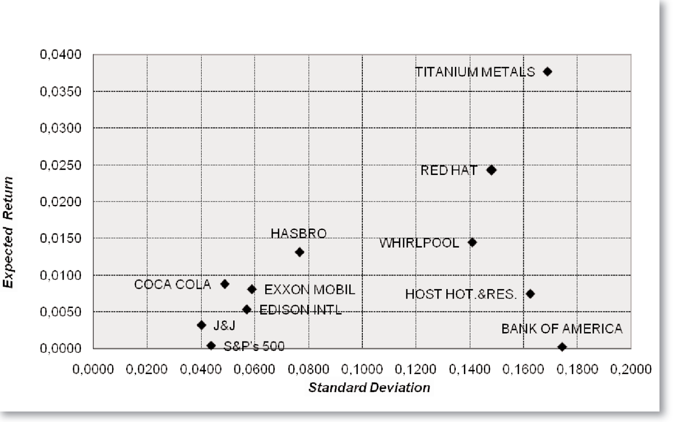
Stocks	HASBRO	COCA COLA	JOHNSON & JOHNSON	HOST HOTELS & RESORTS	EXXON MOBIL	EDISON INTL	WHIRL POOL	TITANIUM METALS	BANK OF AMERICA CORP.	RED HAT
HASBRO	1	0,22682	0,43263	0,40189	0,26119	0,36333	0,43928	0,16477	0,40169	0,2157
COCA COLA	0,22682	1	0,59225	0,21254	0,3422	0,30675	0,32261	0,20941	0,40302	0,10778
JOHNSON & JOHNSON	0,43263	0,59225	1	0,31631	0,1994	0,43671	0,51424	0,11092	0,54078	0,17752
HOST HOTELS & RESORTS	0,40189	0,21254	0,31631	1	0,05089	0,24354	0,66269	0,43855	0,5436	0,20169
EXXON MOBIL	0,26119	0,3422	0,1994	0,05089	1	0,35082	-0,08879	0,26819	0,04437	0,0487
EDISON INTL	0,36333	0,30675	0,43671	0,24354	0,35082	1	0,2619	0,26589	0,30695	0,2956
WHIRLPOOL	0,43928	0,32261	0,51424	0,66269	-0,08879	0,2619	1	0,20745	0,67379	0,22426
TITANIUM METALS	0,16477	0,20941	0,11092	0,43855	0,26819	0,26589	0,20745	1	0,20345	0,27941
BANK OF AMERICA CORP.	0,40169	0,40302	0,54078	0,5436	0,04437	0,30695	0,67379	0,20345	1	0,20276
RED HAT	0,2157	0,10778	0,17752	0,20169	0,0487	0,2956	0,22426	0,27941	0,20276	1

We choose now 5 of the 10 initial assets in order to determine the best efficient frontier. There are different approaches that we can use in order to select the best assets. One of these is to take the assets with the highest Sharp Ratio. Another way we can use is to take the assets that are less risky than the others, so with a lower variance, fixing a constant level of expected return and vice versa. We use the first approach, so we calculate the Sharpe Ratios for each stock.

So let's initially see an expected return - standard deviation plot of the ten risky assets. We can see the level of risk associated to the expected return for each asset. We can say for example, that is much better investing in the 'Titanium Metals' asset than in the 'Bank of America' asset because the first asset has a bigger expected return and a lower risk represented by the standard deviation.

Figure 1.

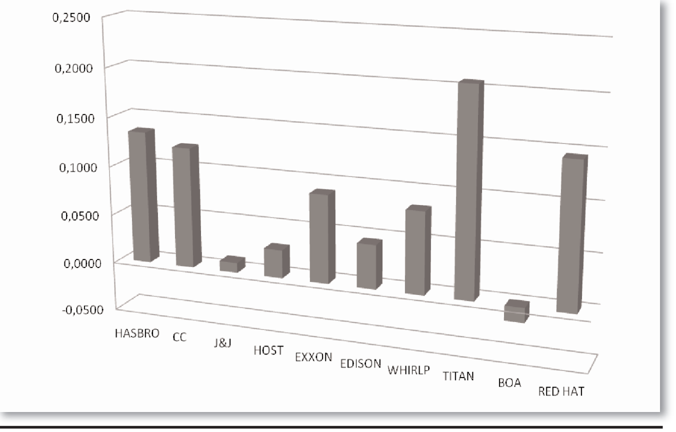
Risk-return plot



We can evaluate better the asset performances in the Sharp Ratio histogram. This is the Sharp ratio formula: $SharpeRatio = \frac{\bar{R} - rf}{\sigma}$, where $\bar{R}$ *rf* and *σ* are the expected return, the risk free rate and the standard deviation. We can say by the Sharp Ratio histogram (figure 2) that the best five risky assets are: Hasbro, Coca Cola, Exxon, Titanium Metals and Red Hat. So these are my five selected assets.

Figure 2.

Sharpe ratio histogram.



Now we determine the efficient frontier for these five assets without imposing any short selling constraint. So the problem is to minimize risk for a given level of excpected return.

$$\min_{\omega} \boldsymbol{\omega}'\boldsymbol{\Sigma}\boldsymbol{\omega}$$
$$s.t.\ \boldsymbol{\omega}'\boldsymbol{\mu} = \mu_p$$
$$\boldsymbol{\omega}'\mathbf{1} = 1$$

,where ω is the portfolio weights vector, Σ is the variance-covariance matrix and μ is the returns vector. The solution is:

$$\omega_* = \frac{C\mu_P - B}{AC - B^2}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \frac{A - B\mu_P}{AC - B^2}\boldsymbol{\Sigma}^{-1}\mathbf{i}$$
$$A = \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} \quad B = \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\mathbf{i} \quad C = \mathbf{i}'\boldsymbol{\Sigma}^{-1}\mathbf{i}$$
$$\omega_* = D + E\mu_P = \frac{A\boldsymbol{\Sigma}^{-1}\mathbf{i} - B\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{AC - B^2} + \frac{C\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - B\boldsymbol{\Sigma}^{-1}\mathbf{i}}{AC - B^2}\mu_P$$

, where **i** is the unitary vector. We can see the component values in the table below.

Table 5. Computed parameters.

	COMPONENTS
A	0,0887
B	5,4851
C	592,972
D	22,511

The equation of the Efficient Frontier will be:

$$\sigma_p = \sqrt{\frac{C\mu_p^2 - 2B\mu_p + A}{AC - B^2}}$$

These are the first seven points that we use in order to draw the efficient frontier:

Table 6. Seven efficient frontier points.

Efficient Frontier Points	
Return	***Standard Deviation***
-0,05000	0,306852198
-0,04900	0,301766812
-0,04800	0,296683043
-0,04700	0,291600976
-0,04600	0,286520701
-0,04500	0,281442315
-0,04400	0,276365923

The Global Minimum Variance (GMV) portfolio is a fully-invested portfolio with the minimum volatility value. As mentioned before, the volatility can be estimated by the standard deviation. The GMV portfolio belongs to efficient frontier and is located on its left end. These will be the parameters for the GMV portfolio.

$$\omega_V = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{i}}{\mathbf{i}'\boldsymbol{\Sigma}^{-1}\mathbf{i}} \qquad \sigma_V = \frac{1}{\sqrt{C}} \qquad \mu_V = \frac{B}{C}$$

The Tangent Portfolio combines this optimal combination of risky assets with a risk-free asset. It has the highest Sharp Ratio. These will be the parameters for the Tangent Portfolio.

$$\omega_E = \frac{\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{\mathbf{i}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}} \qquad \sigma_V = \frac{\sqrt{A}}{B} \qquad \mu_V = \frac{A}{B}$$

We have in the table 7 the corresponding weights for each asset for both the Tangent and the Global Minimum Variance portfolio. We have calculated the portfolio standard deviation and return in each case.

Table 7. GMV and Tangent Portfolio parameters.

Portfolio	GMV	Tangent
HASBRO	*0,141211*	*0,227764*
COCA COLA	*0,535853*	*0,399318*
EXXON M.	*0,300541*	*0,098452*
TITANIUM M.	*-0,021347*	*0,164878*
RED HAT	*0,043742*	*0,109589*
Sum of Weights	*1*	*1*
Standard deviation	*0,04106602*	*0,05429774*
Return	*0,00925018*	*0,01617144*

We determine now the expected return and the standard deviation of the equally weighted portfolio generated with the 5 selected stocks and the equally weighted portfolio generated with the 10 stocks. An equally weighted portfolio would have the same amount of money invested in each unique stock. Therefore, the number of shares of each stock would be different, with more shares of cheaper stocks. An equally weighted portfolio would have to be rebalanced more frequently to maintain equal weight, because stocks prices would diverge quickly. We have in the table below the portfolio standard deviation and return for the two equally weighted portfolios.

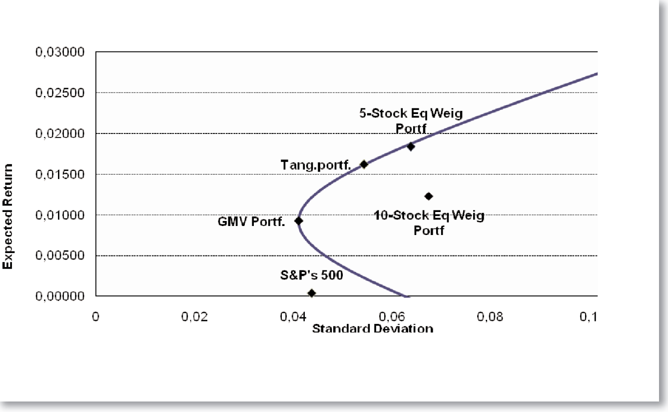
Table 8. Return and standard deviation for the two equally weighted portfolios.

Portfolio	5-Stock Equally Weighted	10-Stock Equally Weighted
Standard deviation	*0,06381338*	*0,067447*
Return	*0,01838952*	*0,012266*

We have in figure 3 the graphical representation of the Efficient Frontier and other relevant portfolios. The Efficient Frontier includes all the efficient portfolios. There are no portfolios with the same standard deviation and a greater return and vice versa. All the rational agents will choose their portfolio in this curve (tangency point with the indifference curves). The Market Index Portfolio is as we expected on the left side of the efficient frontier. The Market Index Portfolio is composed by 500 risky assets including the our five assets of the efficient frontier. As we can see by the graph we can obtain a greater expected return than the S&P portfolio's one without changing the level of standard deviation. We can do this by moving up vertically the Index Portfolio until we reach the Efficient Frontier. So we can say that it exists an Efficient Frontier portfolio more efficient that the Index Portfolio. We reach the same conclusion for the two equally weighted portfolios. So, the two equally weighted portfolios have a lower expected return than the efficient frontier portfolios with the same standard deviation.

Figure 3.

Efficient frontier and some relevant portfolios.



Now we will see the same portfolio compositions imposing the short selling constraint. In finance, short selling (also known as shorting or going short) is the practice of selling assets, usually securities, that have been borrowed from a third part (usually a broker) with the intention of buying identical assets back at a later date to return to the lender. So, the problem in this case is:

$$\min_{\omega} \boldsymbol{\omega}'\boldsymbol{\Sigma}\boldsymbol{\omega}$$
$$s.t.\ \boldsymbol{\omega}'\boldsymbol{\mu} = \mu_p$$
$$\boldsymbol{\omega}'\mathbf{1} = 1$$
$$\boldsymbol{\omega} \geq \mathbf{0}$$

We use the excel solver in order to draw the efficient frontier with the short selling constraint. We have used 23 points and we have in the table below 7 of them.

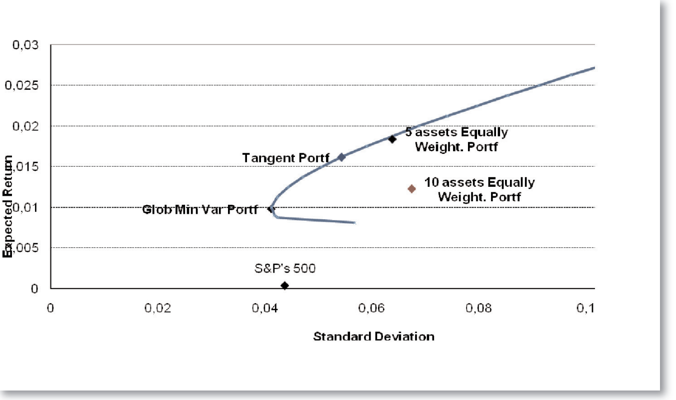
Table 8. Efficient frontier points.

Stocks	HASBRO	COCA COLA	JOHNSON	HOST HOTELS	EXXON MOBIL
HASBRO	*0*	*0,05088*	*0,10271*	*0,14934*	*0,16623*
COCA COLA	*0,05137*	*0,58407*	*0,56742*	*0,52218*	*0,49638*
EXXON MOBIL	*0,94863*	*0,36505*	*0,32986*	*0,27947*	*0,24212*
TITANIUM METALS	*0*	*0*	*0*	*0*	*0,03249*
RED HAT	*0*	*0*	*0*	*0,04901*	*0,06278*
Sum of Weights	*1*	*1*	*1*	*1*	*1*
Optimal Portfolio Return	*0,0081*	*0,00875*	*0,009001*	*0,009999*	*0,011251*
Target Portfolio Return	*0,0081*	*0,00875*	*0,009*	*0,01*	*0,01125*
Optimal Portfolio Variance	*0,00322*	*0,001798*	*0,001747*	*0,001701*	*0,001792*
Portfolio Standard Deviation	*0,056741*	*0,042399*	*0,041794*	*0,041247*	*0,04233*

We have can see in figure 4 the Efficient Frontier, the Tangent Portfolio, the Global Minimum Variance portfolio, the two equally weighted portfolios and the Market Portfolio. So we reach the same conlusion regarding the index portfolio and the two equally weighted portfolios. We can obtain a greater return with the same level of risk on the efficient frontier portfolios.

Figure 4.

Efficient frontier and some relevant portfolios (short selling constraint).



Let's determine now the Efficient Frontier with the risk free asset and the Tangent Portfolio. We use the *US INTERBANK (1 MTH) interest rates* time series in order to have an approximation for the risk free rate. We can compute the risk free rate using:

$$RF = \frac{\sum (y_t / 1200)}{n}$$

, where *y* is the *US INT.(1 Month)* time series and *n* is the number of observations.

The agent optimal choice in this case will be related to its risk aversion coefficient. The optimal portfolio will include a risk-free investment and a risky investment with weights in the risky assets proportional to the risky assets weights in the Tangency Portfolio. The agent problem is:

$$\max_{\omega_P} E[R_P] - \frac{R_A}{2} Var\left[R_P\right]$$

$$s.t. \quad R_P = \sum_{i=1}^{N+1} \omega_i R_i \qquad \sum_{i=1}^{N+1} \omega_i = 1$$

,where *i=1* identifies the risk-free investment. Solving this problem, we can find the optimal weights.

$$\boldsymbol{\omega} = \frac{1}{R_A} \boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\mu} - r_f\right) \frac{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\mu} - r_f\right)}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\mu} - r_f\right)}$$

$$\boldsymbol{\omega} = \frac{1}{R_A} \mathbf{1}'\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\mu} - r_f\right) \frac{\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\mu} - r_f\right)}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\mu} - r_f\right)} = \frac{1}{R_A}\left(B - Ar_f\right)\boldsymbol{\omega}_M$$

The Efficient Frontier equation is

$$\sigma_p = \frac{\mu - RF}{\sqrt{A - B * RF - 1\Sigma^{-1}\mu' * RF + C * RF^2}}$$

, where *A, B, C* are defined as before. So in this case, the Efficient Frontier is a straight line, no more a curve.

We define the weights of the Tangency Portfolio

$$w = \frac{\Sigma^{-1}\mu' - RF * \Sigma^{-1}1'}{1\Sigma^{-1}\mu' - C * RF}$$

where *'RF'* is the risk free rate computed before. So we can obtain the portfolio standard deviation and return with the usual formulas:

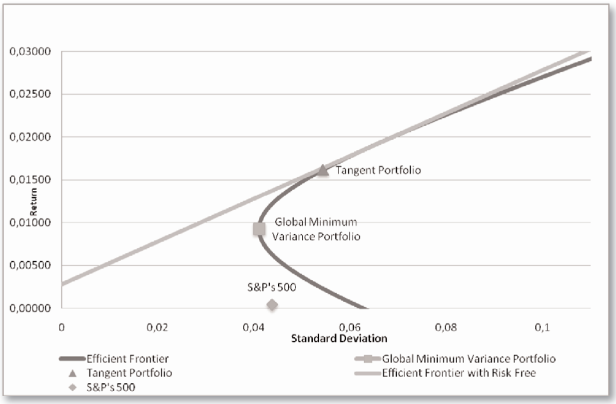
$$\sigma_p = (\omega' \Sigma \omega)^{1/2}$$

$$\mu_p = \omega' \mu$$

We have represented in figure 5 the Efficient Frontier with the risk free asset (the straight line) and the Tangent portfolio. The tangency portfolio in this case is the unique portfolio of the efficient frontier with the risk free asset that does not contain any investment in the risk free asset.

Figure 5.

Efficient frontier with and without the risk free asset (short selling allowed).



We test finally the CAPM (Capital Asset Pricing Model) for the selected stocks. We run a linear regression of the ten stock returns on the market index returns (S&P 500 in the our case).

$$R_i - R_f = \alpha_i + \beta_i (R_m - R_f)$$

Bloomberg adjusts estimated betas with the following formula:

Adjusted beta = 0,66 × Unadjusted beta + 0,34

So these are the estimated betas with the corresponding Bloomberg adjustment:

Table 9. Estimated betas with the corresponding adjustment.

	HASBRO	COCA COLA	J & J	HOST HOT	EXXON M.	EDISON INTL	WHIRL POOL	TITANIUM M.	BANK OF AMERICA	RED HAT
Beta	0,90	0,30	0,42	2,20	0,22	0,67	1,92	1,78	1,79	1,13
Beta adj. (Bloomberg)	0,93	0,54	0,61	1,79	0,48	0,78	1,61	1,51	1,52	1,09

If the beta of an asset is equal to 1, the reaction of the asset return to the market return is proportional. If beta is greater then 1, the reaction to the market return is more than proportional and if beta is less than 1, the asset moves less than proportionally with respect to the market.

We can take for example two extreme cases from the table, ‘Exxon Mobil’ and ‘Host Hotels & Resorts’(see the Bloomberg adjustment). We have beta equal to 0,48 for the ‘Exxon Mobil’ asset, so the reaction of the asset return to the market return is less than proportional. We have an opposite reaction for ‘Host Hotels & Resorts’ which beta is 1,79. We can say in general that we don’t have extreme beta values, so ‘in mean’ the our assets are following the market course. We can see in the next table some statistics for the estimated models. So we have the coefficients estimates with the respective standard errors, the t-statistics and the R square coefficient.

Table 10. Relevant statistics for the estimated models (non significant coefficients in red color, betas in blue color).

	HASBRO		J & J		EXXON M.		WHIRL POOL		BANK OF AMERICA	
Beta/const estim.	0,901	0,013	0,416	0,001	0,216	0,006	1,919	0,016	1,791	0,002
Beta/const std.err.	0,196	0,009	0,107	0,005	0,173	0,008	0,338	0,015	0,465	0,02
Beta/const t test	4,588	1,46	3,874	0,302	1,249	0,765	5,672	1,103	3,849	0,086
R2	0,266		0,206		0,026		0,357		0,203	
	COCA COLA		HOST HOT		EDISON INTL		TITANIUM M.		RED HAT	
Beta/const estim.	0,298	0,007	2,204	0,01	0,666	0,004	1,778	0,039	1,13	0,024
Beta/const std.err.	0,141	0,006	0,391	0,017	0,146	0,006	0,448	0,02	0,417	0,018
Beta/const t test	2,114	1,091	5,639	0,585	4,568	0,652	3,972	2,005	2,707	1,328
R^2	0,072		0,354		0,265		0,214		0,112	

The t statistic is always less than 1.96, so we except in all cases the null hypothesis of α=0. The CAPM equilibrium is verified. The betas are all significant except in one case, the Exxon Mobil asset. We observe discrete values of R square coefficients, so we can say that there is a discrete correlation between asset returns and index returns, in this case the S&P 500.

4. Concluding Remarks

We study in this work some main aspects of Modern Portfolio Theory. We determine the optimal portfolios on a selected stock set and we estimate the Capital Asset Pricing Model. We analyse the stocks of ten international companies, part of the American market. We use the assumptions of MPT in order to minimize portfolio risk (or volatility) for a given amount of expected return, by carefully optimizing the proportions of various assets. So, we reduce our exposure to individual asset risk by holding a diversified portfolio of assets. We analyze the price time series for a five years period from March 31, 2006 to March 31, 2011. This period includes the 2007-2008 US subprime crisis that affected the financial markets of all over the world. We can say the period we have selected for our analysis is characterized by a high volatility and a negative trend caused by adverse economic events.

We observe a relatively negative trend for the ten selected stocks, especially for bank returns. The Petroleum company performs better for this five years period due to the increasing oil price. We choose 5 of the 10 initial assets in order to determine the best efficient frontier. Using the Sharp Ratio we decide that the best five companies are: Global Petroleum, Avon International, Vodafone-Panafon, DHL, Intesa San Paolo Bank. We build the efficient frontier using these five companies. A portfolio lying on the efficient frontier represents the combination offering the minimum possible risk, represented by the standard deviation, for given excepted return. All the rational agents will choose their portfolio in this curve (tangency point with the indifference curves). We reach the same conclusion for the two equally weighted portfolio: they are dominated by the efficient frontier.

Finally, we test the Capital Asset Pricing Model on ten selected stocks. The CAPM gives investors a tool for determining their investment decisions. The empirical test of the CAPM showed that the CAPM was a good tool in predicting the price of individual assets. Although the CAPM was not perfectly accurate, it still provides a legitimate explanation of asset prices, that they're expected return is proportional to their systematic risk and the expected excess return to the market. The inaccuracies in this and other empirical tests can be improved with better proxies for the market and the risk free rate and better econometric techniques.

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